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Note for Intuitionistic HyperRough Set, One–directional S–Hyperrough Set, Tolerance Hyperrough Set, and Dynamic Hyperrough Set

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Abstract

Various set-theoretic models have been proposed to handle uncertainty, including Fuzzy Sets [1], Intuitionistic Fuzzy Sets [2], Neutrosophic Sets [3, 4], and Soft Sets [5, 6]. Rough set theory provides a mathematical framework for approximating subsets using lower and upper bounds defined by equivalence relations, effectively capturing uncertainty in classification and data analysis [7, 8]. Building on these foundational ideas, further generalizations such as Hyperrough Sets and Superhyperrough Sets have been developed. In this paper, we introduce newly defined concepts of the Intuitionistic Hyperrough Set, One–directional S–Hyperrough Set, Tolerance Hyperrough Set, and Dynamic Hyperrough Set. These are extended versions of the Intuitionistic Rough Set, One–directional S–rough Set, Tolerance Rough Set, and Dynamic Rough Set, respectively, constructed using the framework of Hyperrough Sets. Additionally, we explore extensions constructed using the Superhyperrough Set framework.

Keywords: Rough set, Hyperrough set, SuperHyperRough set, Intuitionistic rough set, One-directional S-rough set, Tolerance rough set, Dynamic rough set.

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1|Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets under consideration are assumed to be finite. For further details on each concept and the associated operations, readers are encouraged to consult the relevant references as needed.

1.1|Rough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership [7, 9].

[Set] [10] A *set* is a well-defined collection of distinct elements or objects. If a is an element of a set A, we write $a \in A$; otherwise, we write $a \notin A$.

[Subset] [10] Let A and B be sets. A is called a *subset* of B, denoted $A \subseteq B$, if every element of A is also an element of B. If $A \subseteq B$ but $A \neq B$, then A is called a *proper subset* of B, denoted $A \subset B$.

[Empty Set] [10] The *empty set*, denoted by $\varnothing$, is the unique set containing no elements. Formally, for any set A, $\varnothing \subseteq A$.

[Universal Set] A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

[Rough Set Approximation] [11] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

(1) *Lower Approximation*:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

(2) *Upper Approximation*:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

1.2|Intuitionistic Rough Sets

We define *Intuitionistic Rough Sets* as follows [12, 13, 14].

[Intuitionistic Rough Set] (cf. [12, 14])

Let U be a nonempty universe and let R be an equivalence relation on U, which partitions U into equivalence classes $\{[x]_R : x \in U\}$. For any subset $X \subseteq U$, define its *lower approximation* and *upper approximation* by

$$\underline{X} = \{ x \in U \mid [x]_R \subseteq X \}, \qquad \overline{X} = \{ x \in U \mid [x]_R \cap X \neq \emptyset \}.$$

An *intuitionistic rough set* corresponding to X is the ordered pair

$$\langle \mu_X, \nu_X \rangle,$$

where $\mu_X : U \longrightarrow [0,1]$ $\nu_X : U \longrightarrow [0,1]$ are the membership and non-membership functions, respectively, satisfying

$$0 \leq \mu_X(x) + \nu_X(x) \leq 1 \quad \text{for every } x \in U.$$

The *hesitation margin* (or uncertainty) at x is defined by

$$\pi_X(x) = 1 - \big(\mu_X(x) + \nu_X(x)\big).$$

In the standard construction, μ_X and ν_X are assigned as follows:

(1) If $x \in \underline{X}$, then x belongs entirely to X. Set

$$\mu_X(x) = 1, \qquad \nu_X(x) = 0.$$

(2) If $x \notin \overline{X}$, then the equivalence class $[x]_R$ does not intersect X. In this case, set

$$\mu_X(x) = 0, \qquad \nu_X(x) = 1.$$

(3) If $x \in \overline{X} \setminus \underline{X}$ (the boundary region), assign real values such that

$$0 < \mu_X(x) < 1, \quad 0 < \nu_X(x) < 1, \quad \mu_X(x) + \nu_X(x) < 1.$$

[Example of an Intuitionistic Rough Set] Consider the universe

$$U = \{a, b, c, d\},$$

and suppose that an equivalence relation R partitions U into the following classes:

$$[a]_R = \{a, b\}, \quad [c]_R = \{c\}, \quad [d]_R = \{d\}.$$

Let the set $X \subset U$ be defined as

$$X = \{a, c\}.$$

Then, the lower and upper approximations of X are:

$$\underline{X} = \{\, x \in U \mid [x]_R \subseteq X \,\} = \{c\},$$

$$\overline{X} = \{\, x \in U \mid [x]_R \cap X \neq \emptyset \,\} = \{a, b, c\}.$$

Now, define the membership and non-membership functions μ_X and ν_X as follows:

- For $x \in \underline{X}$ (i.e., $x = c$):

$$\mu_X(c) = 1, \quad \nu_X(c) = 0.$$

- For $x \notin \overline{X}$ (i.e., $x = d$):

$$\mu_X(d) = 0, \quad \nu_X(d) = 1.$$

- For $x \in \overline{X} \setminus \underline{X}$ (i.e., $x = a$ and $x = b$, noting that a and b belong to the same equivalence class $\{a, b\}$): choose

$$\mu_X(a) = \mu_X(b) = 0.5, \quad \nu_X(a) = \nu_X(b) = 0.3.$$

  Then, the hesitation margin for these elements is

$$\pi_X(a) = \pi_X(b) = 1 - (0.5 + 0.3) = 0.2.$$

Thus, the intuitionistic rough set corresponding to X is given by:

$$\langle \mu_X, \nu_X \rangle = \{(a, 0.5, 0.3),\ (b, 0.5, 0.3),\ (c, 1, 0),\ (d, 0, 1)\}.$$

1.3|One-directional S-rough Sets

We define *One-directional S-rough Sets* as follows [15, 16, 17].

[One-directional S-rough Set] [18] Let U be a non-empty universe and let R be an equivalence relation on U with equivalence classes denoted by $[x]$ for each $x \in U$. Let F be a non-empty family of element transfer functions and fix some $f \in F$. For any subset $X \subset U$, define the *f-extension* of X by

$$X_f = \{ u \in U \setminus X \mid f(u) \in X \}.$$

Then the *one-directional S-set* of X is given by

$$X^\circ = X \cup X_f.$$

Next, define the *lower approximation* and *upper approximation* of X° (with respect to the pair (R, F)) by

$$(R, F)_*(X^\circ) = \bigcup \{ [x] \mid x \in U,\ [x] \subseteq X^\circ \},$$

$$(R, F)^\circ(X^\circ) = \bigcup \{ [x] \mid x \in U,\ [x] \cap X^\circ \neq \varnothing \}.$$

The ordered pair

$$\Big((R, F)_*(X^\circ),\ (R, F)^\circ(X^\circ) \Big)$$

is called the *one-directional S-rough set* of X°. Its *boundary region* is defined as

$$Bn_R(X^\circ) = (R, F)^\circ(X^\circ) \setminus (R, F)_*(X^\circ).$$

[A Concrete Example of a One-directional S-rough Set] Consider the universe

$$U = \{1, 2, 3, 4, 5\},$$

with an equivalence relation R that partitions U into the equivalence classes

$$[1] = \{1, 2\}, \quad [3] = \{3, 4\}, \quad [5] = \{5\}.$$

Let

$$X = \{1, 3\}.$$

Define an element transfer function f (from a family F) by specifying

$$f(2) = 1, \quad f(4) \text{ is not defined or does not yield an element in } X.$$

Then the f-extension of X is

$$X_f = \{ u \in U \setminus X \mid f(u) \in X \} = \{2\},$$

so that the one-directional S-set of X is

$$X^\circ = X \cup X_f = \{1, 2, 3\}.$$

Next, we compute the approximations with respect to R:

- The *lower approximation* is

  $$(R, F)_*(X^\circ) = \bigcup \{[x] \mid [x] \subseteq \{1, 2, 3\}\}.$$

  Notice that the equivalence class $[1] = \{1, 2\}$ is fully contained in X°, whereas $[3] = \{3, 4\}$ is not (since $4 \notin X^\circ$). Hence,

  $$(R, F)_*(X^\circ) = [1] = \{1, 2\}.$$

- The *upper approximation* is

  $$(R, F)^\circ(X^\circ) = \bigcup \{[x] \mid [x] \cap \{1, 2, 3\} \neq \varnothing\}.$$

  Both [1] and [3] have a non-empty intersection with X° (since $1 \in [1]$ and $3 \in [3]$); thus,

  $$(R, F)^\circ(X^\circ) = [1] \cup [3] = \{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\}.$$

Therefore, the one-directional S-rough set of X is given by

$$\Big((R,F)_*(X^\circ),\,(R,F)^\circ(X^\circ)\Big) = \big(\{1,2\},\,\{1,2,3,4\}\big),$$

and its boundary region is

$$Bn_R(X^\circ) = \{1,2,3,4\} \setminus \{1,2\} = \{3,4\}.$$

1.4|Tolerance Rough Sets

We define *Tolerance Rough Sets* as follows [19, 20, 21].

[Tolerance Rough Set] [22] Let U be a nonempty universe and let P be a set of attributes. For each attribute $a \in P$, define the similarity measure

$$\mathrm{SIM}_a(x,y) = 1 - \frac{|a(x)-a(y)|}{a_{\max}-a_{\min}},$$

where $a_{\max}$ and $a_{\min}$ denote the maximum and minimum values of a over U, respectively. Given a threshold $\tau \in [0,1]$, we define the tolerance (or similarity) relation $\mathrm{SIM}_{P,\tau}$ on U by

$$(x,y) \in \mathrm{SIM}_{P,\tau} \iff \prod_{a\in P} \mathrm{SIM}_a(x,y) \geq \tau.$$

For any $x \in U$, denote its tolerance class by

$$\mathrm{SIM}_{P,\tau}(x) = \{y \in U \mid (x,y) \in \mathrm{SIM}_{P,\tau}\}.$$

Then, for any subset $X \subseteq U$ the *tolerance lower approximation* of X is defined as

$$P_\tau X = \{x \in U \mid \mathrm{SIM}_{P,\tau}(x) \subseteq X\},$$

and the *tolerance upper approximation* of X is defined as

$$P^\tau X = \{x \in U \mid \mathrm{SIM}_{P,\tau}(x) \cap X \neq \varnothing\}.$$

The ordered pair

$$\langle P_\tau X,\, P^\tau X\rangle$$

is called the *tolerance rough set* of X with respect to the tolerance relation $\mathrm{SIM}_{P,\tau}$.

[A Concrete Example of a Tolerance Rough Set] Assume a universe

$$U = \{x_1, x_2, x_3\},$$

with a single real-valued attribute a (so that $P = \{a\}$). Suppose the attribute values are given by

$$a(x_1) = 1, \quad a(x_2) = 1.2, \quad a(x_3) = 2.$$

Then, $a_{\min} = 1$ and $a_{\max} = 2$. For any $x, y \in U$ the similarity measure becomes

$$\mathrm{SIM}_a(x,y) = 1 - |a(x) - a(y)|.$$

Thus, we obtain:

$$\mathrm{SIM}_a(x_1,x_2) = 1-|1-1.2| = 0.8, \quad \mathrm{SIM}_a(x_1,x_3) = 1-|1-2| = 0, \quad \mathrm{SIM}_a(x_2,x_3) = 1-|1.2-2| = 0.2.$$

Now, choose the threshold $\tau = 0.5$. Then the tolerance relation $\mathrm{SIM}_{\{a\},\tau}$ is defined by

$$(x,y) \in \mathrm{SIM}_{\{a\},\tau} \iff \mathrm{SIM}_a(x,y) \geq 0.5.$$

Hence, the tolerance classes for each element are:

$$\mathrm{SIM}_{\{a\},\tau}(x_1) = \{x_1,x_2\}, \quad \mathrm{SIM}_{\{a\},\tau}(x_2) = \{x_1,x_2\}, \quad \mathrm{SIM}_{\{a\},\tau}(x_3) = \{x_3\}.$$

Let $X = \{x_1\}$. Then the tolerance approximations are computed as follows:

$$P_\tau X = \{x \in U \mid \mathrm{SIM}_{\{a\},\tau}(x) \subseteq \{x_1\}\}.$$

Since for x_1 we have $\mathrm{SIM}_{\{a\},\tau}(x_1) = \{x_1,x_2\} \not\subseteq \{x_1\}$ and similarly for x_2 and x_3, it follows that

$$P_\tau X = \varnothing.$$

On the other hand, the tolerance upper approximation is

$$P^{\tau}X = \{x \in U \mid \mathrm{SIM}_{\{a\},\tau}(x) \cap \{x_1\} \neq \varnothing\}.$$

We have:

$$\mathrm{SIM}_{\{a\},\tau}(x_1) \cap \{x_1\} = \{x_1\} \neq \varnothing, \quad \mathrm{SIM}_{\{a\},\tau}(x_2) \cap \{x_1\} = \{x_1\} \neq \varnothing,$$

and

$$\mathrm{SIM}_{\{a\},\tau}(x_3) \cap \{x_1\} = \varnothing.$$

Thus,

$$P^{\tau}X = \{x_1, x_2\}.$$

The resulting tolerance rough set of X is therefore given by

$$\langle P_{\tau}X,\, P^{\tau}X\rangle = \langle \varnothing,\, \{x_1, x_2\}\rangle.$$

1.5|Dynamic Rough Set

We define *Dynamic Rough Sets* as follows [23, 24, 25].

[Dynamic Rough Set] [23] Let $A = (U, P)$ be an information system where

- U is a nonempty finite set of objects, and
- P is a set of attributes.

Let $X \subseteq U$ be the target (concept) set and let $T \subseteq P$ be a subset of attributes chosen as *dynamic criteria*. Moreover, let d_T^+ and d_T^- be fixed threshold constants (called, respectively, the inward transfer standard and the outward transfer standard).

For each object $x \in U$, we define two coefficients:

$$\rho_T^+(x) = \quad \text{(inward transfer coefficient)}$$
$$\rho_T^-(x) = \quad \text{(outward transfer coefficient)}$$

which quantitatively measure, with respect to the dynamic criteria T, the *potential* for an object currently *not* in X to be included, and for an object *in* X to be excluded, respectively. (In practical applications these coefficients are often computed by aggregating measures on each attribute in T or, in many cases, are simply chosen as constant values for simplicity.)

Then we define:

(1) The **inflated dynamic main set** (or candidate addition set)

$$I_T(X) = \{x \in U \setminus X :\, \rho_T^+(x) \geq d_T^+\}.$$

(2) The **contracted dynamic set** (or candidate removal set)

$$C_T(X) = \{x \in X :\, \rho_T^-(x) \geq d_T^-\}.$$

(3) The resulting **two-direction dynamic set** is defined as

$$D_T(X) = \big(X \cup I_T(X)\big) \setminus C_T(X).$$

Finally, if we take a subset $Q \subseteq P$ of attributes and use it to induce an equivalence relation on U (with equivalence classes denoted by $[x]_Q$), the *dynamic rough approximations* of X are given by

$$\underline{D}_T(X) = \{x \in U :\, [x]_Q \subseteq D_T(X)\} \quad \text{(dynamic lower approximation)}$$
$$\overline{D}_T(X) = \{x \in U :\, [x]_Q \cap D_T(X) \neq \emptyset\} \quad \text{(dynamic upper approximation).}$$

Consider an information system $A = (U, P)$ with

$$U = \{u_1, u_2, u_3, u_4\} \quad \text{and} \quad P = \{p_1, p_2\}.$$

Assume that the initial target set is

$$X = \{u_1, u_2\},$$

and let the dynamic criteria be

$$T = \{p_1\}.$$

Suppose we define the transfer coefficients as follows:

$$\rho_T^+(x) = \begin{cases} 0.8, & \text{if } x \text{ has } p_1 \text{ valued "high"}, \\ 0.4, & \text{otherwise}, \end{cases} \qquad \rho_T^-(x) = \begin{cases} 0.6, & \text{if } x \text{ has } p_1 \text{ valued "low"}, \\ 0.2, & \text{otherwise}. \end{cases}$$

Let the threshold constants be $d_T^+ = 0.7$ and $d_T^- = 0.5$.

Assume that:

- In X, the object u_1 has $p_1 =$ "high" so that $\rho_T^-(u_1) = 0.2$, and u_2 has $p_1 =$ "low" so that $\rho_T^-(u_2) = 0.6$.
- In $U \setminus X$, the objects u_3 and u_4 have $p_1 =$ "high" so that $\rho_T^+(u_3) = \rho_T^+(u_4) = 0.8$.

Then we have:

$$I_T(X) = \{u_3, u_4\} \quad \text{since } 0.8 \geq 0.7,$$
$$C_T(X) = \{u_2\} \quad \text{since } 0.6 \geq 0.5.$$

Thus, the two-direction dynamic set is computed as

$$D_T(X) = (X \cup I_T(X)) \setminus C_T(X) = \big(\{u_1, u_2\} \cup \{u_3, u_4\}\big) \setminus \{u_2\} = \{u_1, u_3, u_4\}.$$

Now, suppose that the attribute set $Q = P$ induces the following partition on U:

$$[u_1]_Q = [u_3]_Q = \{u_1, u_3\} \quad \text{and} \quad [u_2]_Q = [u_4]_Q = \{u_2, u_4\}.$$

Then the dynamic lower approximation is

$$\underline{D}_T(X) = \{x \in U : [x]_Q \subseteq D_T(X)\} = \{u_1, u_3\},$$

because for every $x \in \{u_1, u_3\}$, the entire equivalence class $[x]_Q = \{u_1, u_3\}$ is contained in $D_T(X)$, while for $x \in \{u_2, u_4\}$ the equivalence class $\{u_2, u_4\}$ is not fully contained in $D_T(X)$ (since $u_2 \notin D_T(X)$).

Similarly, the dynamic upper approximation is

$$\overline{D}_T(X) = \{x \in U : [x]_Q \cap D_T(X) \neq \emptyset\} = U,$$

since every equivalence class (whether $\{u_1, u_3\}$ or $\{u_2, u_4\}$) has a non-empty intersection with $D_T(X)$.

This example illustrates how the dynamic rough set extends the classical rough set by allowing the expansion (through $I_T(X)$) and contraction (through $C_T(X)$) of the set X based on dynamic criteria.

1.6|HyperRough Set and SuperHyperRough Set

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [26, 27, 28, 29].

[HyperRough Set] [26, 27] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \emptyset$, then $\underline{F(a)} = \overline{F(a)} = \emptyset$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

[HyperRough Set] Let

$$X = \{u_1, u_2, u_3, u_4, u_5, u_6\}$$

be a nonempty finite universe. Consider two attributes:

$$T_1 : \text{Color with domain } J_1 = \{\text{Red}, \text{Blue}\},$$

$$T_2 : \text{Size with domain } J_2 = \{\text{Small}, \text{Large}\}.$$

The Cartesian product of the attribute domains is

$$J = J_1 \times J_2 = \{(\text{Red}, \text{Small}), (\text{Red}, \text{Large}), (\text{Blue}, \text{Small}), (\text{Blue}, \text{Large})\}.$$

Define the mapping $F : J \to \mathcal{P}(X)$ by

$$\begin{aligned} F(\text{Red}, \text{Small}) &= \{u_1, u_2\}, \\ F(\text{Red}, \text{Large}) &= \{u_3, u_4\}, \\ F(\text{Blue}, \text{Small}) &= \{u_2, u_5\}, \\ F(\text{Blue}, \text{Large}) &= \{u_4, u_6\}. \end{aligned}$$

Assume that an equivalence relation $R \subseteq X \times X$ is defined by the partition

$$[u_1]_R = \{u_1, u_2\}, \quad [u_3]_R = \{u_3, u_4\}, \quad [u_5]_R = \{u_5, u_6\}.$$

For each $a \in J$ the rough set approximations are defined as follows:

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \qquad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Computation for $a = (\textbf{Red}, \textbf{Small})$:

Since

$$F(\text{Red}, \text{Small}) = \{u_1, u_2\},$$

consider the equivalence class for any $x \in X$:

- For $x = u_1$ (or u_2), we have $[u_1]_R = \{u_1, u_2\} \subseteq \{u_1, u_2\}$. Thus, $u_1, u_2 \in \underline{F(\text{Red}, \text{Small})}$.
- For any x with $x \in \{u_3, u_4\}$ or $x \in \{u_5, u_6\}$, the entire equivalence class is not a subset of $F(\text{Red}, \text{Small})$.

Therefore,

$$\underline{F(\text{Red}, \text{Small})} = \{u_1, u_2\}.$$

For the upper approximation, note that

- The equivalence class $[u_1]_R = \{u_1, u_2\}$ intersects $F(\text{Red}, \text{Small})$ (in fact, it equals $F(\text{Red}, \text{Small})$).
- The classes $\{u_3, u_4\}$ and $\{u_5, u_6\}$ have no intersection with $F(\text{Red}, \text{Small})$.

Thus,

$$\overline{F(\text{Red}, \text{Small})} = \{u_1, u_2\}.$$

Computation for $a = (\textbf{Blue}, \textbf{Small})$:

Here,

$$F(\text{Blue}, \text{Small}) = \{u_2, u_5\}.$$

Now, consider the equivalence classes:

- For x in $[u_1]_R = \{u_1, u_2\}$: Since $u_2 \in F(\text{Blue}, \text{Small})$ but $u_1 \notin F(\text{Blue}, \text{Small})$, we have $[u_1]_R \nsubseteq F(\text{Blue}, \text{Small})$. Thus, $\{u_1, u_2\} \nsubseteq F(\text{Blue}, \text{Small})$.
- For x in $[u_5]_R = \{u_5, u_6\}$: $u_5 \in F(\text{Blue}, \text{Small})$ but $u_6 \notin F(\text{Blue}, \text{Small})$. Hence, $[u_5]_R \nsubseteq F(\text{Blue}, \text{Small})$.

Therefore, the lower approximation is empty:

$$\underline{F(\text{Blue}, \text{Small})} = \varnothing.$$

For the upper approximation, include every $x \in X$ with $[x]_R \cap F(\text{Blue}, \text{Small}) \neq \varnothing$:

- The class $[u_1]_R = \{u_1, u_2\}$ intersects $F(\text{Blue}, \text{Small})$ (via u_2).
- The class $[u_5]_R = \{u_5, u_6\}$ intersects $F(\text{Blue}, \text{Small})$ (via u_5).

Thus,

$$\overline{F(\text{Blue}, \text{Small})} = \{u_1, u_2, u_5, u_6\}.$$

This example concretely demonstrates a HyperRough Set with explicit computation of the lower and upper approximations for different attribute combinations.

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty[26, 30, 27]. The definition of n-SuperHyperRough Sets is described as follows.

[n-SuperHyperRough Set] [26, 27] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

[n-SuperHyperRough Set] Let

$$X = \{u_1, u_2, u_3, u_4\}$$

be a nonempty finite universe. Consider two attributes:

$$T_1 : \text{Shape with domain } J_1 = \{\text{Circle}, \text{Square}\},$$

$$T_2 : \text{Color with domain } J_2 = \{\text{Red}, \text{Blue}\}.$$

For each attribute, consider its power set:

$$\mathcal{P}(J_1) = \{\varnothing, \{\text{Circle}\}, \{\text{Square}\}, \{\text{Circle}, \text{Square}\}\},$$

$$\mathcal{P}(J_2) = \{\varnothing, \{\text{Red}\}, \{\text{Blue}\}, \{\text{Red}, \text{Blue}\}\}.$$

The set of all possible attribute value combinations is

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Define a mapping $F : J \to \mathcal{P}(X)$ by specifying values on selected elements of J. For instance, let

$$\begin{aligned} F(\{\text{Circle}\}, \{\text{Red}\}) &= \{u_1, u_2\}, \\ F(\{\text{Square}\}, \{\text{Blue}\}) &= \{u_3\}, \\ F(\{\text{Circle}, \text{Square}\}, \{\text{Red}, \text{Blue}\}) &= X, \\ F(A) &= \varnothing \quad \text{for all other } A \in J. \end{aligned}$$

Assume an equivalence relation $R \subseteq X \times X$ defined by the partition

$$[u_1]_R = \{u_1, u_2\}, \quad [u_3]_R = \{u_3, u_4\}.$$

For any $A \in J$ the rough approximations are defined as:

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \varnothing\}.$$

Computation for $A = (\{\textbf{Circle}\}, \{\textbf{Red}\})$:

Since

$$F(\{\text{Circle}\}, \{\text{Red}\}) = \{u_1, u_2\},$$

we have:

[leftmargin=2em]For $x = u_1$ (or u_2), $[u_1]_R = \{u_1, u_2\} \subseteq \{u_1, u_2\}$. Thus, $u_1, u_2 \in \underline{F(\{\text{Circle}\}, \{\text{Red}\})}$. No other equivalence class is completely contained in $F(\{\text{Circle}\}, \{\text{Red}\})$.

Therefore,

$$\underline{F(\{\text{Circle}\}, \{\text{Red}\})} = \{u_1, u_2\}.$$

For the upper approximation, note:

[leftmargin=2em]The equivalence class $[u_1]_R = \{u_1, u_2\}$ intersects $F(\{\text{Circle}\}, \{\text{Red}\})$.

Thus,

$$\overline{F(\{\text{Circle}\}, \{\text{Red}\})} = \{u_1, u_2\}.$$

Computation for $A = (\{\textbf{Square}\}, \{\textbf{Blue}\})$:

Here,

$$F(\{\text{Square}\}, \{\text{Blue}\}) = \{u_3\}.$$

Now, consider the equivalence class:

[leftmargin=2em]For $x = u_3$ or $x = u_4$, the equivalence class $[u_3]_R = \{u_3, u_4\}$ is not completely contained in $\{u_3\}$ (since $u_4 \notin \{u_3\}$). Hence,

$$\underline{F(\{\text{Square}\}, \{\text{Blue}\})} = \varnothing.$$

However, since $u_3 \in F(\{\text{Square}\}, \{\text{Blue}\})$ and $[u_3]_R$ intersects $F(\{\text{Square}\}, \{\text{Blue}\})$,

$$\overline{F(\{\text{Square}\}, \{\text{Blue}\})} = [u_3]_R = \{u_3, u_4\}.$$

Computation for $A = (\{\textbf{Circle}, \textbf{Square}\}, \{\textbf{Red}, \textbf{Blue}\})$:

Since

$$F(\{\text{Circle}, \text{Square}\}, \{\text{Red}, \text{Blue}\}) = X,$$

we immediately have

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq X\} = X,$$

$$\overline{F(A)} = \{x \in X \mid [x]_R \cap X \neq \varnothing\} = X.$$

This example illustrates an n-SuperHyperRough Set where attribute values are chosen from the power sets of the original domains. The mapping F is defined on $J = \mathcal{P}(J_1) \times \mathcal{P}(J_2)$ and, together with the equivalence relation R, yields concrete lower and upper approximations.

2|Results of This Paper

This section presents the results obtained in this paper.

2.1|Intuitionistic Hyperrough Set

Let X be a nonempty finite universe and let $J = J_1 \times J_2 \times \cdots \times J_n$ be the set of all attribute value combinations corresponding to the attributes $T_1, \ldots, T_n$. Let R be an equivalence relation on X. An *Intuitionistic Hyperrough Set* over X is a pair (F, J) where

$$F : J \to \mathcal{I}(X)$$

is a mapping that assigns to each attribute combination $a = (a_1, a_2, \ldots, a_n) \in J$ an intuitionistic set

$$F(a) = \Big\{ \langle x,\, \mu_F(a)(x),\, \nu_F(a)(x) \rangle \mid x \in X \Big\}$$

with the property that for each $a \in J$ we define the *intuitionistic lower* and *upper approximations* by:

$$\underline{F}(a) = \Big\{ x \in X \mid \forall\, y \in [x]_R,\ \mu_F(a)(y) = 1 \text{ and } \nu_F(a)(y) = 0 \Big\},$$

$$\overline{F}(a) = \Big\{ x \in X \mid [x]_R \cap \{ y \in X \mid \mu_F(a)(y) > 0 \} \neq \varnothing \Big\}.$$

In this way, the pair

$$\Big(\underline{F}(a), \overline{F}(a) \Big)$$

serves as the intuitionistic rough approximation of the intuitionistic set $F(a)$. Notice that by construction,

$$\underline{F}(a) \subseteq \overline{F}(a),$$

and if for each x we have the *crisp condition* $\mu_F(a)(x) \in \{0, 1\}$ (and hence $\nu_F(a)(x) = 1 - \mu_F(a)(x)$), the pair of approximations coincide with the classical rough approximations of $F(a)$.

The following theorem establishes that Intuitionistic Hyperrough Sets generalize both Hyperrough Sets and Intuitionistic Rough Sets.

[Generalization by Intuitionistic Hyperrough Sets] Let X, J, R, and F be as in Definition . Then:

- (i) If, for every $a \in J$, the mapping $F(a)$ is *crisp* (that is, for every $x \in X$, $\mu_F(a)(x) \in \{0, 1\}$ and $\nu_F(a)(x) = 1 - \mu_F(a)(x)$), then the Intuitionistic Hyperrough Set (F, J) coincides with the classical Hyperrough Set.
- (ii) If $n = 1$ (i.e. J reduces to J_1), then (F, J) becomes an Intuitionistic Rough Set.

Proof: **(i)** Suppose that for every $a \in J$ and $x \in X$ we have $\mu_F(a)(x) \in \{0, 1\}$ and $\nu_F(a)(x) = 1 - \mu_F(a)(x)$. Define

$$F'(a) = \{ x \in X \mid \mu_F(a)(x) = 1 \}.$$

Then the intuitionistic lower approximation

$$\underline{F}(a) = \{ x \in X \mid \forall\, y \in [x]_R,\ \mu_F(a)(y) = 1 \}$$

is exactly the same as the classical lower approximation

$$\underline{F'}(a) = \{ x \in X \mid [x]_R \subseteq F'(a) \},$$

and similarly for the upper approximation. Hence, the pair

$$\Big(\underline{F}(a), \overline{F}(a) \Big)$$

recovers the classical rough approximations assigned by the Hyperrough Set (F', J). Thus, Intuitionistic Hyperrough Sets generalize Hyperrough Sets.

(ii) If $n = 1$, then $J = J_1$ and the mapping reduces to

$$F : J_1 \to \mathcal{I}(X).$$

In this case, for each attribute value $a \in J_1$, the pair

$$\left(\underline{F}(a), \overline{F}(a)\right)$$

defines the intuitionistic rough approximations of the intuitionistic set $F(a)$. This exactly matches the standard definition of an Intuitionistic Rough Set defined on X. Hence, the model reduces appropriately. □

2.2|Intuitionistic Superhyperrough Sets

We define *Intuitionistic Superhyperrough Sets* as follows.

[Intuitionistic Superhyperrough Set] Let X be a nonempty finite universe and let

$$J^* = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$$

be the Cartesian product of the power sets of the attribute domains J_i. Let R be an equivalence relation on X. An *Intuitionistic Superhyperrough Set* over X is a pair (G, J^*) where

$$G : J^* \to \mathcal{I}(X)$$

is a mapping that assigns to each attribute set combination

$$A = (A_1, A_2, \ldots, A_n) \in J^* \quad (A_i \subseteq J_i)$$

an intuitionistic set

$$G(A) = \left\{\langle x,\, \mu_G(A)(x),\, \nu_G(A)(x)\rangle \mid x \in X\right\}.$$

For each $A \in J^*$, define the intuitionistic lower and upper approximations by

$$\underline{G}(A) = \left\{x \in X \mid \forall\, y \in [x]_R,\ \mu_G(A)(y) = 1 \text{ and } \nu_G(A)(y) = 0\right\},$$

$$\overline{G}(A) = \left\{x \in X \mid [x]_R \cap \{y \in X \mid \mu_G(A)(y) > 0\} \neq \varnothing\right\}.$$

Thus, the pair

$$\left(\underline{G}(A), \overline{G}(A)\right)$$

is the intuitionistic rough approximation of the intuitionistic set $G(A)$.

The following theorem shows that Intuitionistic Superhyperrough Sets generalize both SuperHyperrough Sets and Intuitionistic Hyperrough Sets.

[Generalization by Intuitionistic Superhyperrough Sets] Let X, J^*, R, and G be as in Definition . Then:

(i) If for every $A \in J^*$ the mapping $G(A)$ is crisp (that is, for every $x \in X$, $\mu_G(A)(x) \in \{0, 1\}$ and $\nu_G(A)(x) = 1 - \mu_G(A)(x)$), then (G, J^*) coincides with a classical n-SuperHyperrough Set.

(ii) If the mapping G is restricted to the case where each component A_i is a singleton (so that J^* is effectively reduced to the Cartesian product $J_1 \times \cdots \times J_n$), then (G, J^*) reduces to an Intuitionistic Hyperrough Set.

Proof: **(i)** Assume that for all $A \in J^*$ and for all $x \in X$ the membership and nonmembership functions satisfy $\mu_G(A)(x) \in \{0, 1\}$ and $\nu_G(A)(x) = 1 - \mu_G(A)(x)$. Define

$$G'(A) = \{x \in X \mid \mu_G(A)(x) = 1\}.$$

Then the intuitionistic approximations

$$\underline{G}(A) = \{x \in X \mid [x]_R \subseteq G'(A)\} \quad \text{and} \quad \overline{G}(A) = \{x \in X \mid [x]_R \cap G'(A) \neq \varnothing\}$$

coincide with the lower and upper approximations of a classical SuperHyperrough Set (G', J^*).

(ii) Now suppose that G is restricted to those $A = (A_1, A_2, \ldots, A_n) \in J^*$ for which each A_i is a singleton, i.e., $A_i = \{a_i\}$ for some $a_i \in J_i$. In this case, the set J^* is isomorphic to the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Then the mapping G restricted to this subdomain is exactly a mapping

$$F : J \to \mathcal{I}(X),$$

as in Definition . Therefore, the approximations $\underline{G}(A)$ and $\overline{G}(A)$ coincide with those defined by F, and the model reduces to an Intuitionistic Hyperrough Set. This shows that Intuitionistic Superhyperrough Sets generalize Intuitionistic Hyperrough Sets. □

2.3|One–directional S–Hyperrough Set

We define *One–directional S–Hyperrough Set* as follows.

[One–directional S–Hyperrough Set] Let X, R and J be as above. Suppose a mapping

$$F : J \to \mathcal{P}(X)$$

is given. For each $a \in J$, define its one–directional S–extension by

$$F(a)^{\circ} = F(a) \cup \{\, u \in X \setminus F(a) \mid f(u) \in F(a) \,\}.$$

Then, the *lower* and *upper approximations* of $F(a)^{\circ}$ are defined as

$$\underline{F^{\circ}}(a) = \{x \in X \mid [x]_R \subseteq F(a)^{\circ}\},$$

$$\overline{F^{\circ}}(a) = \{x \in X \mid [x]_R \cap F(a)^{\circ} \neq \varnothing\}.$$

The pair

$$\big(\underline{F^{\circ}}(a),\, \overline{F^{\circ}}(a)\big)$$

is the intuitionistic rough approximation of $F(a)^{\circ}$ for each $a \in J$. The collection

$$\{\, (\underline{F^{\circ}}(a),\, \overline{F^{\circ}}(a)) : a \in J \,\}$$

is called a *One–directional S–Hyperrough Set* over X.

[One–directional S–Hyperrough Set] Let

$$X = \{1, 2, 3, 4, 5\},$$

and let the equivalence relation R partition X as follows:

$$[1]_R = \{1, 2\}, \quad [3]_R = \{3, 4\}, \quad [5]_R = \{5\}.$$

Suppose the attribute domain is

$$J = \{A, B\},$$

and define the mapping $F : J \to \mathcal{P}(X)$ by

$$F(A) = \{1, 3, 5\} \quad \text{and} \quad F(B) = \{2, 4\}.$$

Let the element transfer function $f : X \to X$ be given by

$$f(2) = 1, \quad f(4) = 3, \quad \text{and } f(x) = x \text{ for } x \in \{1, 3, 5\}.$$

Then the one–directional S–extension is defined by

$$F(a)^{\circ} = F(a) \cup \{\, u \in X \setminus F(a) \mid f(u) \in F(a) \,\} \quad \text{for } a \in J.$$

For $a = A$:

$$F(A) = \{1, 3, 5\}.$$

The elements not in $F(A)$ are $X \setminus F(A) = \{2, 4\}$. Since

$$f(2) = 1 \in F(A) \quad \text{and} \quad f(4) = 3 \in F(A),$$

we have

$$F(A)^{\circ} = \{1, 3, 5\} \cup \{2, 4\} = X.$$

For $a = B$:

$$F(B) = \{2, 4\}.$$

Now, $X \setminus F(B) = \{1, 3, 5\}$ and for each $u \in \{1, 3, 5\}$ we find

$$f(1) = 1 \notin \{2, 4\}, \quad f(3) = 3 \notin \{2, 4\}, \quad f(5) = 5 \notin \{2, 4\}.$$

Thus,

$$F(B)^\circ = F(B) = \{2,4\}.$$

Next, we form the rough approximations using the equivalence relation R.

For $a = A$**:** Since $F(A)^\circ = X$, every equivalence class is contained in X; hence,

$$\underline{F^\circ}(A) = \{x \in X \mid [x]_R \subseteq X\} = X,$$

and

$$\overline{F^\circ}(A) = \{x \in X \mid [x]_R \cap X \neq \emptyset\} = X.$$

For $a = B$**:** We have $F(B)^\circ = \{2,4\}$. Then the lower approximation is

$$\underline{F^\circ}(B) = \{x \in X \mid [x]_R \subseteq \{2,4\}\}.$$

Notice:

$$[1]_R = \{1,2\} \not\subseteq \{2,4\}, \quad [3]_R = \{3,4\} \not\subseteq \{2,4\}, \quad [5]_R = \{5\} \not\subseteq \{2,4\}.$$

Thus,

$$\underline{F^\circ}(B) = \varnothing.$$

The upper approximation is

$$\overline{F^\circ}(B) = \{x \in X \mid [x]_R \cap \{2,4\} \neq \emptyset\}.$$

Here,

$$[1]_R \cap \{2,4\} = \{2\} \neq \varnothing, \quad [3]_R \cap \{2,4\} = \{4\} \neq \varnothing, \quad [5]_R \cap \{2,4\} = \varnothing.$$

So,

$$\overline{F^\circ}(B) = \{1,2,3,4\}.$$

Thus, the One–directional S–Hyperrough Set is given by:

$$\{(\underline{F^\circ}(A), \overline{F^\circ}(A)), (\underline{F^\circ}(B), \overline{F^\circ}(B))\} = \{(X,X), (\varnothing, \{1,2,3,4\})\}.$$

[Generalization Property of One–directional S–Hyperrough Sets] Let $F : J \to \mathcal{P}(X)$ be as in Definition .

(i) If the transfer function f is trivial (i.e., if $f(u) = u$ for all $u \in X$, so that $F(a)^\circ = F(a)$ for all $a \in J$), then the One–directional S–Hyperrough Set

$$\{\, (\underline{F^\circ}(a), \overline{F^\circ}(a)) : a \in J \,\}$$

coincides with the classical Hyperrough Set defined by F.

(ii) If $n = 1$ so that $J = J_1$, then the model reduces to the one–directional S–rough set.

Proof: **(i)** Assume that f is trivial, that is, $f(u) = u$ for every $u \in X$. Then for any $F(a) \subseteq X$ we have

$$F(a)^\circ = F(a) \cup \{u \in X \setminus F(a) \mid f(u) \in F(a)\} = F(a) \cup \{u \in X \setminus F(a) \mid u \in F(a)\} = F(a).$$

Consequently, the approximations become

$$\underline{F^\circ}(a) = \{x \in X \mid [x]_R \subseteq F(a)\}$$

and

$$\overline{F^\circ}(a) = \{x \in X \mid [x]_R \cap F(a) \neq \varnothing\},$$

which are exactly the classical rough set approximations for $F(a)$. Hence, the One–directional S–Hyperrough Set reduces to the classical Hyperrough Set.

(ii) If $n = 1$, then J is simply J_1 and the mapping

$$F : J_1 \to \mathcal{P}(X)$$

defines for each attribute value $a \in J_1$ a subset $F(a) \subseteq X$. The one–directional S–extension $F(a)^\circ$ is then used to form the lower and upper approximations exactly as in the definition of a one–directional S–rough set. Thus, the One–directional S–Hyperrough Set (which in this case is indexed by the single attribute domain J_1) coincides with the one–directional S–rough set. □

2.4|One–directional S–SuperHyperrough Set

We define *One–directional S–SuperHyperrough Set* as follows.

[One–directional S–Superhyperrough Set] Let X be a nonempty finite universe, and let

$$J^* = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$$

be the Cartesian product of the power sets of the attribute domains. Suppose that a mapping

$$G : J^* \to \mathcal{P}(X)$$

is given. For each $A = (A_1, A_2, \ldots, A_n) \in J^*$, define its one–directional S–extension by

$$G(A)^\circ = G(A) \cup \{\, u \in X \setminus G(A) \mid f(u) \in G(A) \,\}.$$

Then the *lower* and *upper approximations* of $G(A)^\circ$ are defined as

$$\underline{G^\circ}(A) = \{x \in X \mid [x]_R \subseteq G(A)^\circ\},$$

$$\overline{G^\circ}(A) = \{x \in X \mid [x]_R \cap G(A)^\circ \neq \varnothing\}.$$

The collection

$$\{\, (\underline{G^\circ}(A), \overline{G^\circ}(A)) : A \in J^* \,\}$$

is called a *One–directional S–Superhyperrough Set* over X.

[One–directional S–Superhyperrough Set] Let

$$X = \{1, 2, 3, 4\},$$

and let the equivalence relation R partition X as

$$[1]_R = \{1, 2\} \quad \text{and} \quad [3]_R = \{3, 4\}.$$

Consider a single attribute T_1 with domain

$$J_1 = \{\text{red}, \text{blue}\}.$$

Then, the power set is

$$J^* = \mathcal{P}(J_1) = \{\varnothing, \{\text{red}\}, \{\text{blue}\}, \{\text{red}, \text{blue}\}\}.$$

Define the mapping $G : J^* \to \mathcal{P}(X)$ by

$$G(\varnothing) = \varnothing, \quad G(\{\text{red}\}) = \{1, 3\}, \quad G(\{\text{blue}\}) = \{2, 4\}, \quad G(\{\text{red}, \text{blue}\}) = X.$$

Let the transfer function $f : X \to X$ be given by

$$f(2) = 1, \quad f(4) = 3, \quad f(x) = x \text{ for } x \in \{1, 3\}.$$

Then, for each $A \in J^*$ the one–directional S–extension is defined by

$$G(A)^\circ = G(A) \cup \{\, u \in X \setminus G(A) \mid f(u) \in G(A)\}.$$

We compute each case:

Case 1. $A = \{\textbf{red}\}$**:** $G(\{\text{red}\}) = \{1, 3\}$. Then,

$$X \setminus G(\{\text{red}\}) = \{2, 4\}.$$

Since $f(2) = 1 \in \{1, 3\}$ and $f(4) = 3 \in \{1, 3\}$, we obtain

$$G(\{\text{red}\})^\circ = \{1, 3\} \cup \{2, 4\} = X.$$

Thus, the rough approximations are:

$$\underline{G^\circ}(\{\text{red}\}) = \{x \in X \mid [x]_R \subseteq X\} = X,$$

$$\overline{G^\circ}(\{\text{red}\}) = X.$$

Case 2. $A = \{\textbf{blue}\}$**:** $G(\{\text{blue}\}) = \{2, 4\}$. Then,

$$X \setminus G(\{\text{blue}\}) = \{1, 3\}.$$

For $u \in \{1,3\}$, we have $f(1) = 1 \notin \{2,4\}$ and $f(3) = 3 \notin \{2,4\}$. Hence,

$$G(\{\text{blue}\})^\circ = \{2,4\}.$$

The lower approximation is

$$\underline{G^\circ}(\{\text{blue}\}) = \{x \in X \mid [x]_R \subseteq \{2,4\}\}.$$

Since $[1]_R = \{1,2\} \not\subseteq \{2,4\}$ and $[3]_R = \{3,4\} \not\subseteq \{2,4\}$, it follows that

$$\underline{G^\circ}(\{\text{blue}\}) = \varnothing.$$

The upper approximation is

$$\overline{G^\circ}(\{\text{blue}\}) = \{x \in X \mid [x]_R \cap \{2,4\} \neq \varnothing\}.$$

We have:

$$[1]_R \cap \{2,4\} = \{2\} \neq \varnothing, \quad [3]_R \cap \{2,4\} = \{4\} \neq \varnothing.$$

Thus,

$$\overline{G^\circ}(\{\text{blue}\}) = \{1,2,3,4\} = X.$$

Case 3. $A = \{\textbf{red}, \textbf{blue}\}$**:** By definition,

$$G(\{\text{red}, \text{blue}\}) = X,$$

so that

$$G(\{\text{red}, \text{blue}\})^\circ = X.$$

Thus, both approximations equal X.

Case 4. $A = \varnothing$**:** Here, $G(\varnothing) = \varnothing$. Then clearly,

$$G(\varnothing)^\circ = \varnothing,$$

and the lower and upper approximations are both empty.

Summarizing, the One–directional S–Superhyperrough Set over X is the collection

$$\{\, (\underline{G^\circ}(A), \overline{G^\circ}(A)) : A \in J^* \,\},$$

with

$$\begin{aligned} A = \varnothing : &\quad (\varnothing, \varnothing), \\ A = \{\text{red}\} : &\quad (X, X), \\ A = \{\text{blue}\} : &\quad (\varnothing, X), \\ A = \{\text{red}, \text{blue}\} : &\quad (X, X). \end{aligned}$$

[Generalization Property of One–directional S–Superhyperrough Sets] Let $G : J^* \to \mathcal{P}(X)$ be as in Definition .

(i) If, for every $A \in J^*$, the mapping $G(A)$ is crisp (that is, if for every $x \in X$ we have $x \in G(A)$ or $x \notin G(A)$ so that the one–directional extension is trivial), then the One–directional S–Superhyperrough Set coincides with the classical Superhyperrough Set.

(ii) If the mapping G is restricted to those $A = (A_1, \ldots, A_n) \in J^*$ where every A_i is a singleton (so that J^* is isomorphic to $J = J_1 \times \cdots \times J_n$), then the One–directional S–Superhyperrough Set reduces to a One–directional S–Hyperrough Set.

Proof: **(i)** Suppose that for every $A \in J^*$ the set $G(A)$ is crisp, i.e., for each $x \in X$ there is no uncertainty (the one–directional S–extension is redundant). Then

$$G(A)^\circ = G(A) \cup \{u \in X \setminus G(A) \mid f(u) \in G(A)\} = G(A).$$

Thus, the approximations become

$$\underline{G^\circ}(A) = \{x \in X \mid [x]_R \subseteq G(A)\}$$

and

$$\overline{G^\circ}(A) = \{x \in X \mid [x]_R \cap G(A) \neq \varnothing\}.$$

These are exactly the classical definitions for an n-Superhyperrough Set. Therefore, the One–directional S–Superhyperrough Set generalizes the classical Superhyperrough Set.

(ii) Now assume that G is restricted to those $A = (A_1, \ldots, A_n) \in J^*$ for which every A_i is a singleton. In this case, J^* becomes isomorphic to

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Define the mapping $F : J \to \mathcal{P}(X)$ by identifying $F(a_1, \ldots, a_n) = G(\{a_1\}, \ldots, \{a_n\})$. Then, by Definition , the one–directional S–extension of $F(a_1, \ldots, a_n)$ is

$$F(a)^\circ = F(a) \cup \{u \in X \setminus F(a) \mid f(u) \in F(a)\},$$

and its approximations are given in the same way as in Definition . Hence, the One–directional S–Superhyperrough Set reduces precisely to a One–directional S–Hyperrough Set. □

2.5|Tolerance Hyperrough Set

The classical hyperrough set is defined (for an equivalence relation) as the family of all subsets that yield identical lower and upper approximations. We now extend this idea to the tolerance case.

[Tolerance Hyperrough Set] Let X be a nonempty finite universe, and let

$$J = J_1 \times J_2 \times \cdots \times J_n$$

be the Cartesian product of the attribute domains. Let $F : J \to \mathcal{P}(X)$ be a mapping. For each $a \in J$, define the *tolerance hyperextension* of $F(a)$ as

$$F(a)^\tau := F(a) \quad \text{(if no further extension is desired)},$$

and consider its tolerance approximations with respect to $\mathrm{SIM}_{P,\tau}$:

$$\underline{F_\tau}(a) := \{x \in X \mid \mathrm{SIM}_{P,\tau}(x) \subseteq F(a)\},$$

$$\overline{F^\tau}(a) := \{x \in X \mid \mathrm{SIM}_{P,\tau}(x) \cap F(a) \neq \varnothing\}.$$

Then, for each $a \in J$, the pair

$$\left(\underline{F_\tau}(a), \, \overline{F^\tau}(a)\right)$$

constitutes the tolerance rough approximation of $F(a)$. The collection

$$\{ \, (\underline{F_\tau}(a), \, \overline{F^\tau}(a)) \mid a \in J \, \}$$

is called a *Tolerance Hyperrough Set* over X.

[Generalization of Hyperrough and Tolerance Rough Sets] Let $F : J \to \mathcal{P}(X)$ be as in Definition . Then:

(i) If the tolerance relation becomes crisp (for example, if $\tau = 1$ so that $\mathrm{SIM}_{P,1}$ is equivalent to equality), then the approximations $\underline{F_\tau}(a)$ and $\overline{F^\tau}(a)$ coincide with the classical rough approximations, and the Tolerance Hyperrough Set reduces to the classical Hyperrough Set.

(ii) If the number of attributes is $n = 1$ (so that $J \cong J_1$), then the mapping $F : J \to \mathcal{P}(X)$ yields, for each $a \in J$,

$$(\underline{F_\tau}(a), \, \overline{F^\tau}(a))$$

which is exactly the tolerance rough approximation of $F(a)$. Hence the Tolerance Hyperrough Set reduces to a Tolerance Rough Set.

Proof: **(i)** If $\tau = 1$ then for any $x, y \in X$ we have $\mathrm{SIM}_{P,1}(x, y) = 1$ if and only if $a(x) = a(y)$ for all $a \in P$. In this case, the tolerance relation $\mathrm{SIM}_{P,1}$ is the classical indiscernibility (equivalence) relation and thus

$$\underline{F_\tau}(a) = \{x \in X \mid [x]_R \subseteq F(a)\}$$

and

$$\overline{F^\tau}(a) = \{x \in X \mid [x]_R \cap F(a) \neq \varnothing\},$$

which coincide with the classical hyperrough approximations. Therefore, the Tolerance Hyperrough Set generalizes the classical Hyperrough Set.

(ii) If $n = 1$ then $J = J_1$ and the mapping $F : J_1 \to \mathcal{P}(X)$ assigns to every single attribute value a subset $F(a) \subseteq X$. The tolerance approximations defined by

$$\underline{F_\tau}(a) = \{x \in X \mid \mathrm{SIM}_{P,\tau}(x) \subseteq F(a)\}$$

and

$$\overline{F^\tau}(a) = \{x \in X \mid \mathrm{SIM}_{P,\tau}(x) \cap F(a) \neq \varnothing\}$$

are exactly those given by the standard tolerance rough set model. Hence, the Tolerance Hyperrough Set reduces to a Tolerance Rough Set. □

[Tolerance Hyperrough Set] Let

$$X = \{1, 2, 3, 4, 5\}.$$

Consider two condition attributes with domains

$$J_1 = \{\mathrm{Red}, \mathrm{Blue}\} \quad \text{and} \quad J_2 = \{\mathrm{High}, \mathrm{Low}\}.$$

Then, the Cartesian product is

$$J = J_1 \times J_2 = \{(\mathrm{Red}, \mathrm{High}),\ (\mathrm{Red}, \mathrm{Low}),\ (\mathrm{Blue}, \mathrm{High}),\ (\mathrm{Blue}, \mathrm{Low})\}.$$

Define the mapping $F : J \to \mathcal{P}(X)$ by

$$F(\mathrm{Red}, \mathrm{High}) = \{1, 2, 3\}, \quad F(\mathrm{Red}, \mathrm{Low}) = \{3, 4, 5\},$$

and for simplicity we let

$$F(\mathrm{Blue}, \mathrm{High}) = \{1, 2, 3\}, \quad F(\mathrm{Blue}, \mathrm{Low}) = \{3, 4, 5\}.$$

Assume a tolerance relation on X specified as follows. For each $x \in X$, define

$$SIM(x) = \begin{cases} \{1, 2\} & \text{if } x = 1, \\ \{1, 2, 3\} & \text{if } x = 2, \\ \{2, 3, 4\} & \text{if } x = 3, \\ \{3, 4, 5\} & \text{if } x = 4, \\ \{4, 5\} & \text{if } x = 5. \end{cases}$$

Then, for any subset $S \subseteq X$,

$$P_\tau S = \{x \in X \mid SIM(x) \subseteq S\} \quad \text{and} \quad P^\tau S = \{x \in X \mid SIM(x) \cap S \neq \varnothing\}.$$

Case: Consider the attribute combination $(\mathrm{Red}, \mathrm{High})$. Then

$$F(\mathrm{Red}, \mathrm{High}) = \{1, 2, 3\}.$$

Lower approximation:

- For $x = 1$: $SIM(1) = \{1, 2\} \subseteq \{1, 2, 3\}$ so $1 \in P_\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 2$: $SIM(2) = \{1, 2, 3\} \subseteq \{1, 2, 3\}$ so $2 \in P_\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 3$: $SIM(3) = \{2, 3, 4\}$ but $4 \notin \{1, 2, 3\}$; hence $3 \notin P_\tau F(\mathrm{Red}, \mathrm{High})$.
- $x = 4$ and $x = 5$ are not in $F(\mathrm{Red}, \mathrm{High})$ so we do not include them.

Thus,

$$P_\tau F(\mathrm{Red}, \mathrm{High}) = \{1, 2\}.$$

Upper approximation:

- For $x = 1$: $SIM(1) = \{1, 2\}$ intersects $\{1, 2, 3\}$ (in fact, $\{1, 2\} \neq \varnothing$); so $1 \in P^\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 2$: $SIM(2) = \{1, 2, 3\}$ clearly intersects $\{1, 2, 3\}$; so $2 \in P^\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 3$: $SIM(3) = \{2, 3, 4\}$ intersects $\{1, 2, 3\}$ (since $2, 3 \in \{1, 2, 3\}$); so $3 \in P^\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 4$: $SIM(4) = \{3, 4, 5\}$ intersects $\{1, 2, 3\}$ (since $3 \in \{1, 2, 3\}$); so $4 \in P^\tau F(\mathrm{Red}, \mathrm{High})$.
- For $x = 5$: $SIM(5) = \{4, 5\}$ does not intersect $\{1, 2, 3\}$; so $5 \notin P^\tau F(\mathrm{Red}, \mathrm{High})$.

Thus,

$$P^{\tau}F(\text{Red}, \text{High}) = \{1, 2, 3, 4\}.$$

Therefore, the Tolerance Hyperrough Set corresponding to the attribute combination (Red, High) is the pair

$$\big(P_{\tau}F(\text{Red}, \text{High}),\ P^{\tau}F(\text{Red}, \text{High})\big) = (\{1, 2\},\ \{1, 2, 3, 4\}).$$

2.6|Tolerance Superhyperrough Sets

Classically, a superhyperrough set is defined as the collection of maximal elements (with respect to set inclusion) of a hyperrough set. We generalize this notion as follows.

[Tolerance Superhyperrough Set] Let X be a nonempty finite universe, and let

$$J^* = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$$

be the Cartesian product of the power sets of the attribute domains. Let $G : J^* \to \mathcal{P}(X)$ be a mapping. For each $A = (A_1, \ldots, A_n) \in J^*$, define the tolerance approximations by

$$\underline{G_{\tau}}(A) := \{x \in X \mid \text{SIM}_{P,\tau}(x) \subseteq G(A)\},$$

$$\overline{G^{\tau}}(A) := \{x \in X \mid \text{SIM}_{P,\tau}(x) \cap G(A) \neq \varnothing\}.$$

Then the pair

$$\Big(\underline{G_{\tau}}(A),\, \overline{G^{\tau}}(A)\Big)$$

represents the tolerance rough approximation of $G(A)$. The family

$$\{\, (\underline{G_{\tau}}(A),\, \overline{G^{\tau}}(A)) \mid A \in J^* \,\}$$

is called a *Tolerance Superhyperrough Set* over X.

[Generalization of Superhyperrough and Tolerance Hyperrough Sets] Let $G : J^* \to \mathcal{P}(X)$ be as in Definition . Then:

(i) If for every $A \in J^*$ the mapping $G(A)$ is crisp (that is, the tolerance approximations yield no uncertainty), then the tolerance approximations $\underline{G_{\tau}}(A)$ and $\overline{G^{\tau}}(A)$ coincide with the classical rough approximations. In this case, the Tolerance Superhyperrough Set is equivalent to the classical Superhyperrough Set.

(ii) If G is restricted to those elements $A = (A_1, \ldots, A_n) \in J^*$ where each A_i is a singleton (so that J^* is isomorphic to $J = J_1 \times \cdots \times J_n$), then the tolerance approximations reduce to those given in Definition ; that is, the Tolerance Superhyperrough Set reduces to a Tolerance Hyperrough Set.

Proof: **(i)** If $G(A)$ is crisp for every $A \in J^*$ then for every $x \in X$ we have that $x \in G(A)$ or $x \notin G(A)$. In this situation, the tolerance approximations simplify to

$$\underline{G_{\tau}}(A) = \{x \in X \mid [x]_{\text{SIM}_{P,1}} \subseteq G(A)\},$$

and

$$\overline{G^{\tau}}(A) = \{x \in X \mid [x]_{\text{SIM}_{P,1}} \cap G(A) \neq \varnothing\},$$

which are exactly those of the classical n-Superhyperrough Set. (In essence, when there is no additional uncertainty induced by the tolerance relation, the model is classical.)

(ii) Suppose now that G is restricted to those $A = (A_1, \ldots, A_n) \in J^*$ in which every A_i is a singleton. Then J^* is naturally isomorphic to

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Define a mapping $F : J \to \mathcal{P}(X)$ by

$$F(a_1, \ldots, a_n) = G(\{a_1\}, \ldots, \{a_n\}).$$

Then for every $a \in J$ the tolerance approximations for $F(a)$ are precisely those defined in Definition . Thus, under this restriction, the Tolerance Superhyperrough Set reduces to a Tolerance Hyperrough Set. □

[Tolerance Superhyperrough Set] Let

$$X = \{1, 2, 3, 4\}.$$

Consider a single attribute T_1 with domain

$$J_1 = \{\text{red}, \text{blue}\}.$$

Then, the power set of the domain is

$$\mathcal{P}(J_1) = \{\varnothing, \{\text{red}\}, \{\text{blue}\}, \{\text{red}, \text{blue}\}\}.$$

Define a mapping $G : \mathcal{P}(J_1) \to \mathcal{P}(X)$ by

$$G(\varnothing) = \varnothing, \quad G(\{\text{red}\}) = \{1, 3\}, \quad G(\{\text{blue}\}) = \{2, 4\}, \quad G(\{\text{red}, \text{blue}\}) = X.$$

Assume a tolerance relation on X given by:

$$SIM(1) = \{1, 2\}, \quad SIM(2) = \{1, 2, 3\}, \quad SIM(3) = \{2, 3, 4\}, \quad SIM(4) = \{3, 4\}.$$

Then for any $S \subseteq X$,

$$P_\tau S = \{x \in X \mid SIM(x) \subseteq S\} \quad \text{and} \quad P^\tau S = \{x \in X \mid SIM(x) \cap S \neq \varnothing\}.$$

We now compute the tolerance approximations for several cases.

Case 1. $A = \varnothing$. Then, $G(\varnothing) = \varnothing$. Hence,

$$P_\tau G(\varnothing) = \varnothing \quad \text{and} \quad P^\tau G(\varnothing) = \varnothing.$$

Case 2. $A = \{\text{red}\}$. Then, $G(\{\text{red}\}) = \{1, 3\}$.

Lower approximation:
Check each $x \in X$:

- For $x = 1$: $SIM(1) = \{1, 2\}$. Since $\{1, 2\} \not\subseteq \{1, 3\}$ (because $2 \notin \{1, 3\}$), $1 \notin P_\tau G(\{\text{red}\})$.
- For $x = 2$: $SIM(2) = \{1, 2, 3\}$ is not a subset of $\{1, 3\}$ (as $2 \notin \{1, 3\}$).
- For $x = 3$: $SIM(3) = \{2, 3, 4\}$ is not contained in $\{1, 3\}$ (since 2 and 4 are missing).
- For $x = 4$: $SIM(4) = \{3, 4\}$ is not contained in $\{1, 3\}$ (since $4 \notin \{1, 3\}$).

Thus,

$$P_\tau G(\{\text{red}\}) = \varnothing.$$

Upper approximation:

- $x = 1$: $SIM(1) = \{1, 2\}$ and $\{1, 2\} \cap \{1, 3\} = \{1\} \neq \varnothing$; so $1 \in P^\tau G(\{\text{red}\})$.
- $x = 2$: $SIM(2) = \{1, 2, 3\}$ intersects $\{1, 3\}$ (since both 1 and 3 are in the intersection); so $2 \in P^\tau G(\{\text{red}\})$.
- $x = 3$: $SIM(3) = \{2, 3, 4\}$ intersects $\{1, 3\}$ (via element 3); so $3 \in P^\tau G(\{\text{red}\})$.
- $x = 4$: $SIM(4) = \{3, 4\}$ intersects $\{1, 3\}$ (since $3 \in \{1, 3\}$); so $4 \in P^\tau G(\{\text{red}\})$.

Thus,

$$P^\tau G(\{\text{red}\}) = \{1, 2, 3, 4\}.$$

Case 3. $A = \{\text{red}, \text{blue}\}$. Then, $G(\{\text{red}, \text{blue}\}) = X = \{1, 2, 3, 4\}$. Hence, both approximations are

$$P_\tau G(\{\text{red}, \text{blue}\}) = X \quad \text{and} \quad P^\tau G(\{\text{red}, \text{blue}\}) = X.$$

Thus, the Tolerance Superhyperrough Set is the collection of pairs:

$$\begin{aligned} A = \varnothing : &\quad (\varnothing, \varnothing), \\ A = \{\text{red}\} : &\quad (\varnothing, \{1,2,3,4\}), \\ A = \{\text{blue}\} : &\quad (\varnothing, \{1,2,3,4\}), \\ A = \{\text{red}, \text{blue}\} : &\quad (X, X). \end{aligned}$$

2.7|Dynamic Hyperrough Set

We define *Dynamic Hyperrough Set* as follows.

[Dynamic Hyperrough Set] Let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Assume there is a mapping

$$F : J \to \mathcal{P}(X)$$

that assigns to each attribute combination $a = (a_1, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$. In order to capture dynamic effects, assume that for every $a \in J$ there exist two transfer functions

$$\rho_a^+ : X \to [0,1] \quad \text{and} \quad \rho_a^- : X \to [0,1],$$

together with fixed thresholds d_a^+, $d_a^- \in [0,1]$. Then for each $a \in J$ define the *inward (addition) set* and the *outward (removal) set* by

$$\begin{aligned} I(a) &= \{x \in X \setminus F(a) : \rho_a^+(x) \geq d_a^+\}, \\ C(a) &= \{x \in F(a) : \rho_a^-(x) \geq d_a^-\}. \end{aligned}$$

The *dynamic modified set* associated with a is

$$D(a) = \big(F(a) \cup I(a)\big) \setminus C(a).$$

The pair (D, J), where $D : J \to \mathcal{P}(X)$ is given by $a \mapsto D(a)$, is called a *dynamic Hyperrough Set*. Its dynamic approximations are given by

$$\underline{D(a)} = \{x \in X : [x]_R \subseteq D(a)\} \quad \text{and} \quad \overline{D(a)} = \{x \in X : [x]_R \cap D(a) \neq \emptyset\}.$$

[Dynamic Hyperrough Set] Let

$$X = \{u_1,\, u_2,\, u_3,\, u_4,\, u_5\}$$

be a finite universe. Consider two attributes:

$$T_1 : \text{Color with domain } J_1 = \{\text{Red}, \text{Blue}\},$$

$$T_2 : \text{Shape with domain } J_2 = \{\text{Circle}, \text{Square}\}.$$

Then the attribute combination domain is

$$J = J_1 \times J_2.$$

Define the mapping

$$F : J \to \mathcal{P}(X)$$

by setting, for instance,

$$\begin{aligned} F(\text{Red}, \text{Circle}) = \{u_1, u_2\}, &\quad F(\text{Red}, \text{Square}) = \{u_3\}, \\ F(\text{Blue}, \text{Circle}) = \{u_2, u_4\}, &\quad F(\text{Blue}, \text{Square}) = \{u_5\}. \end{aligned}$$

To incorporate dynamic effects, for each attribute combination $a \in J$ we introduce two transfer functions:

$$\rho_a^+ : X \to [0,1] \quad \text{(inward transfer)},$$

$$\rho_a^- : X \to [0,1] \quad \text{(outward transfer)}.$$

For the combination $a = (\text{Red}, \text{Circle})$, suppose that the dynamic transfer values are assigned as follows:

$$\rho^+_{(\text{Red},\text{Circle})}(x) = \begin{cases} 0.8, & \text{if } x \in \{u_3, u_4\}, \\ 0.5, & \text{otherwise}, \end{cases}$$

$$\rho^{-}_{(\text{Red,Circle})}(x) = \begin{cases} 0.6, & \text{if } x \in \{u_2\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Let the thresholds be

$$d^{+}_{(\text{Red,Circle})} = 0.7 \quad \text{and} \quad d^{-}_{(\text{Red,Circle})} = 0.5.$$

Then we define the *inward (addition) set* and the *outward (removal) set* for $a = (\text{Red}, \text{Circle})$ by

$$I(a) = \{x \in X \setminus F(a) : \rho^{+}_{a}(x) \geq d^{+}_{a}\} = \{u_3, u_4\},$$

$$C(a) = \{x \in F(a) : \rho^{-}_{a}(x) \geq d^{-}_{a}\} = \{u_2\}.$$

Hence, the dynamic modified set is computed as

$$D(a) = \big(F(a) \cup I(a)\big) \setminus C(a) = \big(\{u_1, u_2\} \cup \{u_3, u_4\}\big) \setminus \{u_2\} = \{u_1, u_3, u_4\}.$$

Now, assume that an equivalence relation $R \subseteq X \times X$ is given by partitioning

$$X \quad \text{into} \quad [u_1]_R = \{u_1, u_3, u_4\} \quad \text{and} \quad [u_2]_R = \{u_2, u_5\}.$$

Then the dynamic lower and upper approximations for the combination $(\text{Red}, \text{Circle})$ are

$$\underline{D(a)} = \{x \in X : [x]_R \subseteq D(a)\} = \{u_1, u_3, u_4\},$$

$$\overline{D(a)} = \{x \in X : [x]_R \cap D(a) \neq \emptyset\} = X.$$

Thus, this example concretely illustrates the computation of a Dynamic Hyperrough Set.

[Dynamic Hyperrough Set Generalizes Hyperrough Set and Dynamic Rough Set] The dynamic Hyperrough Set (D, J) defined in Definition generalizes:

(1) The classical Hyperrough Set: If for every $a \in J$ and for every $x \in X$ the dynamic conditions are vacuous, i.e.

$$\rho^{+}_{a}(x) < d^{+}_{a} \quad \text{and} \quad \rho^{-}_{a}(x) < d^{-}_{a},$$

then $I(a) = \varnothing$ and $C(a) = \varnothing$, so that $D(a) = F(a)$.

(2) The dynamic Rough Set: In the special case when $n = 1$ (so that $J = J_1$), the definition reduces to that of the dynamic Rough Set.

Proof: **(1) Recovery of the Classical Hyperrough Set:** Assume that for all $a \in J$ and every $x \in X$, the inequalities

$$\rho^{+}_{a}(x) < d^{+}_{a} \quad \text{and} \quad \rho^{-}_{a}(x) < d^{-}_{a}$$

hold. Then by definition,

$$I(a) = \{x \in X \setminus F(a) : \rho^{+}_{a}(x) \geq d^{+}_{a}\} = \varnothing,$$

and

$$C(a) = \{x \in F(a) : \rho^{-}_{a}(x) \geq d^{-}_{a}\} = \varnothing.$$

Thus,

$$D(a) = (F(a) \cup \varnothing) \setminus \varnothing = F(a),$$

for every $a \in J$. Hence, the dynamic construction yields exactly the classical Hyperrough Set.

(2) Recovery of the Dynamic Rough Set: When $n = 1$, the Cartesian product reduces to $J = J_1$ and the mapping $F : J_1 \to \mathcal{P}(X)$ assigns subsets of X based on a single attribute value. In this case, the dynamic procedure of forming

$$D(a) = \big(F(a) \cup I(a)\big) \setminus C(a)$$

is exactly that adopted in the dynamic Rough Set framework. Therefore, the dynamic Hyperrough Set generalizes the dynamic Rough Set when there is only one attribute. □

2.8|Dynamic SuperHyperrough Set

We define *Dynamic SuperHyperrough Set* as follows.

[Dynamic SuperHyperrough Set] Let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Assume there is a mapping

$$F : J \to \mathcal{P}(X)$$

assigning to each combination

$$A = (A_1, A_2, \ldots, A_n) \quad \text{with } A_i \subseteq J_i,$$

a subset $F(A) \subseteq X$. In order to incorporate dynamics, suppose that for every $A \in J$ there exist dynamic transfer functions

$$\rho_A^+ : X \to [0,1] \quad \text{and} \quad \rho_A^- : X \to [0,1],$$

with thresholds d_A^+, $d_A^- \in [0,1]$. Then define

$$I(A) = \{x \in X \setminus F(A) : \rho_A^+(x) \geq d_A^+\}, \quad C(A) = \{x \in F(A) : \rho_A^-(x) \geq d_A^-\}.$$

The *dynamic modified set* for A is given by

$$D(A) = \Big(F(A) \cup I(A)\Big) \setminus C(A).$$

Thus, the pair (D, J) with $D : J \to \mathcal{P}(X)$ defined by $A \mapsto D(A)$ is called an *Dynamic SuperHyperrough Set*. Its lower and upper approximations are defined as

$$\underline{D(A)} = \{x \in X : [x]_R \subseteq D(A)\}, \quad \overline{D(A)} = \{x \in X : [x]_R \cap D(A) \neq \emptyset\}.$$

[Dynamic SuperHyperrough Set] Let

$$X = \{u_1,\, u_2,\, u_3,\, u_4\}$$

be a finite universe. Consider two attributes:

$$T_1 : \text{Type with domain } J_1 = \{A, B\},$$

$$T_2 : \text{Category with domain } J_2 = \{X, Y\}.$$

For each attribute, we consider its power set. Hence, define

$$\mathcal{P}(J_1) = \{\varnothing, \{A\}, \{B\}, \{A, B\}\}, \quad \mathcal{P}(J_2) = \{\varnothing, \{X\}, \{Y\}, \{X, Y\}\}.$$

Then, the attribute combination domain is given by

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Define a mapping

$$F : J \to \mathcal{P}(X)$$

as follows:

$$F(\{A\}, \{X\}) = \{u_1, u_2\}, \quad F(\{B\}, \{X\}) = \{u_3\},$$
$$F(\{A\}, \{Y\}) = \{u_2, u_4\}, \quad F(\{A, B\}, \{X, Y\}) = X,$$

and for all other combinations, let

$$F(A) = \varnothing.$$

To introduce dynamic modifications, assign for each combination $A \in J$ appropriate transfer functions. For instance, for the combination

$$A = (\{A\}, \{X\}),$$

assume that

$$\rho_A^+(x) = \begin{cases} 0.9, & \text{if } x \in \{u_3, u_4\}, \\ 0.4, & \text{otherwise,} \end{cases}$$

and

$$\rho_A^-(x) = \begin{cases} 0.7, & \text{if } x \in \{u_2\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Choose thresholds

$$d_A^+ = 0.8 \quad \text{and} \quad d_A^- = 0.6.$$

Then, for $A = (\{A\}, \{X\})$, we compute:

$$I(A) = \{x \in X \setminus F(A) : \rho_A^+(x) \geq d_A^+\} = \{u_3, u_4\},$$
$$C(A) = \{x \in F(A) : \rho_A^-(x) \geq d_A^-\} = \{u_2\}.$$

Thus, the dynamic modified set for A is

$$D(A) = \big(F(A) \cup I(A)\big) \setminus C(A) = \big(\{u_1, u_2\} \cup \{u_3, u_4\}\big) \setminus \{u_2\} = \{u_1, u_3, u_4\}.$$

Suppose now that the equivalence relation R on X partitions the universe as

$$[u_1]_R = \{u_1, u_3\} \quad \text{and} \quad [u_2]_R = \{u_2, u_4\}.$$

Then the dynamic approximations for $A = (\{A\}, \{X\})$ are given by

$$\underline{D(A)} = \{x \in X : [x]_R \subseteq D(A)\} = \{u_1, u_3\},$$
$$\overline{D(A)} = \{x \in X : [x]_R \cap D(A) \neq \emptyset\} = X.$$

This example concretely demonstrates the computation within a Dynamic SuperHyperrough Set, where the domain of attribute combinations is expanded to include subsets of the original attribute domains.

[Dynamic SuperHyperrough Set Generalizes SuperHyperrough Set and Dynamic Hyperrough Set] The dynamic SuperHyperrough Set (D, J) defined in Definition generalizes:

(1) The classical SuperHyperrough Set: If for every $A \in J$ and every $x \in X$,

$$\rho_A^+(x) < d_A^+ \quad \text{and} \quad \rho_A^-(x) < d_A^-,$$

then $I(A) = \varnothing$ and $C(A) = \varnothing$, so that $D(A) = F(A)$.

(2) The dynamic Hyperrough Set: If we restrict J to the subset

$$J' = \{(\{a_1\}, \{a_2\}, \ldots, \{a_n\}) : a_i \in J_i\},$$

then the mapping F on J' and the corresponding dynamic modification coincide with the ones in Definition ; hence, the dynamic SuperHyperrough Set reduces to the dynamic Hyperrough Set.

Proof: **(1) Recovery of the Classical SuperHyperrough Set:** Assume that for all $A \in J$ and every $x \in X$ the conditions

$$\rho_A^+(x) < d_A^+ \quad \text{and} \quad \rho_A^-(x) < d_A^-$$

are satisfied. Then, by definition,

$$I(A) = \varnothing \quad \text{and} \quad C(A) = \varnothing.$$

Thus,

$$D(A) = \big(F(A) \cup \varnothing\big) \setminus \varnothing = F(A).$$

It follows that the dynamic modification disappears, and the structure coincides with that of a classical SuperHyperrough Set.

(2) Reduction to the Dynamic Hyperrough Set: Now, consider the subset

$$J' = \{(\{a_1\}, \{a_2\}, \ldots, \{a_n\}) : a_i \in J_i\} \subseteq J.$$

Define the mapping F' on J' by

$$F'\big((\{a_1\}, \{a_2\}, \ldots, \{a_n\})\big) := F(a_1, a_2, \ldots, a_n).$$

For each element of J', the dynamic transfer functions and thresholds (restricted to singleton sets) are taken to be identical to those used in the dynamic Hyperrough Set (Definition). Hence, for every $A = (\{a_1\}, \ldots, \{a_n\}) \in J'$ one obtains

$$D(A) = \big(F'(A) \cup I(A)\big) \setminus C(A)$$

which is exactly the same as in the dynamic Hyperrough Set definition. Therefore, under the restriction to singleton components, the dynamic SuperHyperrough Set reduces to the dynamic Hyperrough Set. □

3|Conclusion and Future Tasks

In this paper, we introduced newly defined concepts of the Intuitionistic Hyperrough Set, One–Directional S–Hyperrough Set, Tolerance Hyperrough Set, and Dynamic Hyperrough Set. In future work, we aim to explore extensions of these frameworks to Graphs [31], Hypergraphs [32, 33], and SuperHypergraphs [34, 35].

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

Competing interests

Author has declared that no competing interests exist.

Consent to Publish declaration

The author approved to Publish declarations.

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